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# On existence of continuous selection for finite sets

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**Abstract**

We prove a theorem which allows, under suitable conditions, to extend a Vietoris continuous selection for two-point subsets of a Hausdorff space $X$ to a Vietoris continuous selection on all finite subsets of $X$. In particular, such an extension is always possible if $X$ has at most one non-isolated point. We then apply this extension theorem to obtain the following result: If $X$ is a scattered hereditarily paracompact Hausdorff space which has a Vietoris continuous selection for two-point subsets of $X$, then $X$ also has a Vietoris continuous selection for all finite subsets of $X$. This gives a partial answer to a question of Gutev and Nogura [5].

## 1 Preliminaries

For a topological space $X$ we use $\mathcal{F}(X)$ to denote the set of all non-empty closed subsets of $X$ endowed with the Vietoris topology $\tau_V$. Recall that $\tau_V$ is generated by all collections of the form

$$\langle\mathcal{V}\rangle = \{S \in \mathcal{F}(X) : S \subseteq \bigcup \mathcal{V} \text{ and } S \cap V \neq \emptyset \text{ whenever } V \in \mathcal{V}\},$$

where $\mathcal{V}$ runs over the finite families of open subsets of $X$. Define $\mathcal{K}(X) = \{S \in \mathcal{F}(X) : |S| < \omega\}$ and $\mathcal{F}_n(X) = \{S \in \mathcal{F}(X) : |S| \leq n\}$ for $n \in \omega$.

If $\mathcal{D} \subseteq \mathcal{F}(X)$, then a map $f : \mathcal{D} \to X$ is a *selection* on $\mathcal{D}$ provided that $f(S) \in S$ for every $S \in \mathcal{D}$. A selection $f : \mathcal{D} \to X$ is *continuous* if it is continuous with respect to the relative Vietoris topology on $\mathcal{D}$. A selection $f : \mathcal{F}_2(X) \to X$ on $\mathcal{F}_2(X)$ is called a *weak selection.*

Suppose that $f$ is a weak selection. Then $f$ defines a natural order-like relation "$\preceq$ "on $X$, by letting $x \preceq y$ if and only if $f(\{x, y\}) = x$. For convenience, we will write that $x \prec y$ if $x \preceq y$ and $x \neq y$. If $B$ and $C$ are (not necessarily non-empty) subsets of $X$ , we write $B \prec C$ provided that $y \prec z$ for every $y \in B$ and $z \in C$.

**Proposition 1.1.** [4, Theorem 3.1] *Let $X$ be a Hausdorff space, $f : \mathcal{F}_2(X) \to X$ be a selection, and let $\preceq$ be the order-like relation generated by $f$. Furthermore assume that $x, y \in X$ satisfy $x \prec y$. Then $f$ is continuous at $\{x, y\}$ if and only if there exist disjoint open sets $U$ and $V$ such that $x \in U$, $y \in V$, and $U \preceq V$.*

Given a selection $f$ on $\mathcal{F}_2(X)$, one can define a selection $f^*$ on $\mathcal{F}_2(X)$ as follows: $f^*(\{x\}) = x$ for $x \in X$ and $f^*(\{x, y\}) = y$ if and only if $f(\{x, y\}) = x$ whenever $x, y \in X$, $x \neq y$ [4]. Since $X$ is a Hausdorff space, Proposition 1.1 yields that $f$ is a continuous weak selection if and only if $f^*$ is a continuous weak selection. We will denote by $\preceq^*$ the order-like relation generated by $f^*$. Obviously, $x \preceq y$ if and only if $y \preceq^* x$.

**Definition 1.2.** [3] Given a weak selection $f : \mathcal{F}_2(X) \to X$ and a set $S \in \mathcal{F}(X)$, we will call a subset $B \subseteq S$ an *$f$-maximum* (*$f$-minimum*) of $S$ if $B \in \mathcal{F}(X)$ and the following conditions hold:

(1) $S \setminus B \prec B$ $(B \prec S \setminus B)$,

(2) if $C \subseteq S$, $C \in \mathcal{F}(X)$, and $S \setminus C \prec C$ $(C \prec S \setminus C)$, then $B \subseteq C$.

Clearly, a set $B$ is an $f$-maximum of $S$ if and only if it is an $f^*$-minimum of $S$, so the following proposition is essentially due to Gutev and Nogura [3].

**Proposition 1.3.** *Let $X$ be a Hausdorff space, $f : \mathcal{F}_2(X) \to X$ a weak selection and $\preceq$ the order-like relation generated by $f$. Then every non-empty compact subset $S \in \mathcal{F}(X)$ has an unique $f$-minimum and $f$-maximum.*

**Definition 1.4.** [3] Given a weak selection $f : \mathcal{F}_2(X) \to X$ and a compact (in particular, finite) set $S \in \mathcal{F}(X)$, we will use $\min_f S$ and $\max_f S$ to denote the $f$-minimum and $f$-maximum of $S$, respectively.

**Proposition 1.5.** *Suppose that $f : \mathcal{F}_2(X) \to X$ is a weak selection, $\preceq$ is an order-like relation generated by $f$, $S, D, E \subseteq X$, $S \subseteq D \cup E$, $D \prec E$, $S \cap D \in \mathcal{F}(X)$ and $S$ is compact. Then $\min_f S \subseteq D$.*

# 2 Extension of weak selection to finite sets

The special case of item ($i$) of our next lemma (when $U = X$) has been proved in [3].

**Lemma 2.1.** *Let $X$ be a Hausdorff space, $U$ its open subset, Bd $U$ the boundary of $U$, $f : \mathcal{F}_2(X) \to X$ a continuous weak selection, and $\preceq$ the order-like relation generated by $f$. Let $\mathcal{E}_U = \{S \in \mathcal{K}(X) : S \cap U \neq \emptyset\}$.*

*(i) If $U \prec Bd\, U$, then the map $\varphi : \mathcal{E}_U \to \mathcal{E}_U$ defined by $\varphi(S) = \min_f(S \cap U)$ is continuous.*

*(ii) If $Bd\, U \prec U$, then the map $\psi : \mathcal{E}_U \to \mathcal{E}_U$ defined by $\psi(S) = \max_f(S \cap U)$ is continuous.*

By using Lemma 2.1, we get the following main theorem.

**Theorem 2.2.** *Let $X$ be a Hausdorff space, $f : \mathcal{F}_2(X) \to X$ be a continuous weak selection, and let $\preceq$ be the order-like relation generated by $f$. Furthermore, assume that $p \in X$ and $g$ is a continuous selection on $\mathcal{K}(X \setminus \{p\})$. Then there exists a continuous selection $h$ on $\mathcal{K}(X)$ extending $f$.*

The following is an application of Theorem 2.2.

**Corollary 2.3.** *Let $X$ be a Hausdorff space with a single non-isolated point $p \in X$, and let $f : \mathcal{F}_2(X) \to X$ be a weak selection. Then $f$ can be extended to a continuous selection on $\mathcal{K}(X)$.*

# 3 Selections for finite subsets of paracompact scattered spaces

Let us recall the definition of a scattered space. For every ordinal number $\alpha$, we define by transfinite induction the $\alpha$-derivative of a space $X$: $X^{(0)} = X$; $X^{(\alpha+1)} = (X^{(\alpha)})' = \{x \in X : x \text{ is not an isolated point of } X^{(\alpha)}\}$; $X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)}$ if $\alpha$ is limit. A space $X$ is called scattered if $X^{(\alpha)} = \emptyset$ for some ordinal $\alpha$. For a scattered space $X$, the *height* $h(X)$ of $X$ is the smallest ordinal $\alpha$ such that $X^{(\alpha)} = \emptyset$. For every $\alpha < h(X)$, each $x \in X^{(\alpha)} \setminus X^{(\alpha+1)}$ is an isolated point of $X^{(\alpha)}$, thus there exists a neighborhood $V_x$ of $x$ such that $V_x \cap X^{(\alpha)} = \{x\}$.

The following lemma is part of folklore, for example [2, Lemma 2.1].

**Lemma 3.1.** *Suppose that $\mathcal{U}$ is a clopen partition of a space $X$ such that for every $U \in \mathcal{U}$, there exists a continuous selection $f_U : \mathcal{K}(U) \to U$ on $\mathcal{K}(U)$. Then $\mathcal{K}(X)$ has a continuous selection.*

The existence of continuous selections on scattered spaces has been studied in [1, 2]. A countable regular space has a continuous selection if and only if it is scattered [2, Theorem 2.4]. A paracompact scattered space admits a continuous selection provided that every point has a countable pseudo-base [2, Theorem 2.3]. By using transfinite induction with respect to the height $h(X)$ of $X$ and applying Theorem 2.2, we can get our last theorem which contributes to this topic and provides a positive partial answer to Problem 5 from [5].

**Theorem 3.2.** *If a scattered hereditarily paracompact Hausdorff space $X$ has a weak selection, then it also has a continuous selection on $\mathcal{K}(X)$.*

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# References

[1] G. Artico, U. Marconi, R. Moresco, and J. Pelant, Selectors and scattered spaces, Topology Appl. **111** (2001), 35–48.

[2] S. Fuji, K. Miyazaki, and T. Nogura, *Vietoris continuous selections on scattered spaces*, J. Math. Soc. Japan **54** (2002), no. 2, 273–281.

[3] S. García-Ferreira, V. Gutev, and T. Nogura, *Extension of 2-point selections*, New Zealand J. Math. (2006), accepted for publication.

[4] V. Gutev and T. Nogura, *Selections and order-like relations*, Applied General Topology **2** (2001), 205–218.

[5] V. Gutev and T. Nogura, *Michael's selection problem*, in: Open Problems in Topology, II, to appear.